Illustrative Mathematics

Taking Modeling Seriously in High School

William McCallum

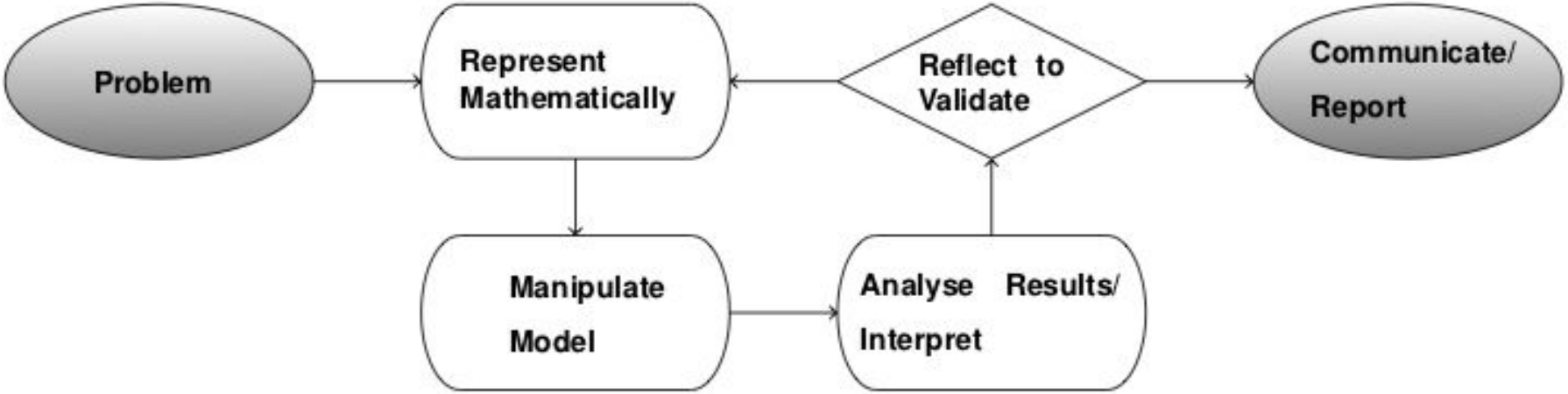
“How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?"

Albert Einstein

What is modeling?

- "Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions." (CCSSM)
- "Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena." (GAIMME)

The modeling cycle



Uses of modeling

. . . medicine, engineering, ecology, weather forecasting, oil exploration, finance and economics, business and marketing, climate modeling, designing search engines, understanding social networks, public key cryptography and cybersecurity, the space program, astronomy and cosmology, biology and genetics, criminology, using genetics to reconstruct how early humans spread over the planet, in testing and designing new drugs, in compressing images (JPEG) and music (MP3), in creating the algorithms that cell phones use to communicate, to optimize air traffic control and schedule flights, to design cars and wind turbines, to recommend which books (Amazon), music (Pandora) and movies (Netflix) an individual might like based on other things they rated highly . . .

MP4. Model with mathematics.

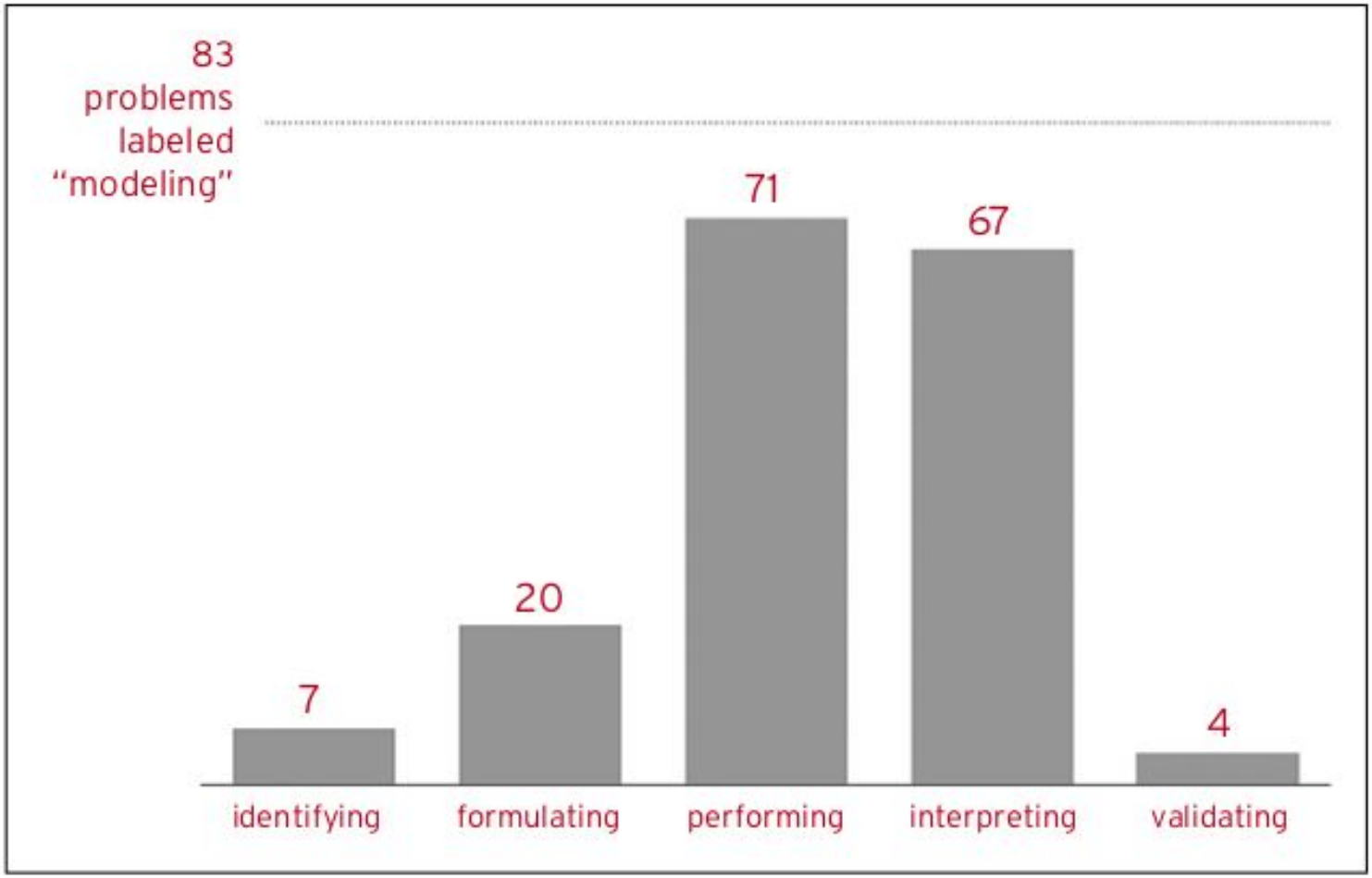
- height above the ground of a person riding a Ferris wheel (F-TF.5)
- how people's heights vary (S-ID.1)
- a risk factor for a disease (S-ID.5)
- the effectiveness of a medical treatment (S-ID.5)
- the amount of money in a savings account to which periodic additions are made (A-SSE.4)
- the unemployment rate (N-Q.1)

Modeling Progression

Driving for gas (GAIMME)

Most drivers have a “usual” region in which they do most of their driving. However, gas prices may vary widely so that gas may be substantially cheaper somewhere other than within that usual region. Would it be more economical to go to a station outside the usual region to buy gas? Thus, the general question we wish to address is, “How might we determine which gas station is the most cost-efficient?”

"Missing the Math" (Dan Meyer)



Mathematics Teacher, 2015

Beyond word problems

- Initially there might not be enough information, the modeler might have to do some research.
- There is room to interpret, formulate, and refine the question.
- A range of reasonable assumptions and answers is possible.
- Modeling requires choices to be made by the modeler.

Achieve framework cognitive complexity

	Level 1	Level 2	Level 3
Procedural Complexity:[13]	Solving the problem entails little procedural[14] demand or procedural demand is below grade level.	Solving the problem entails common or grade-level procedure(s) with friendly numbers.	Solving the problem requires common or grade-level procedure(s) with unfriendly numbers,[15] an unconventional combination of procedures, or requires unusual perseverance or organizational skills in the execution of the procedure(s).
Conceptual Complexity:[16]	Solving the problem requires students to recall or recognize a grade-level concept. The student does not need to relate concepts or demonstrate a line of reasoning.	Students may need to relate multiple grade-level concepts or different types, create multiple representations or solutions, or connect concepts with procedures or strategies. The student must do some reasoning, but may not need to demonstrate a line of reasoning.	Solving the problem requires students to relate multiple grade-level concepts and to evidence reasoning, planning, analysis, judgment, and/or creative thought OR work with a sophisticated (nontypical) line of reasoning.
Application Complexity:	Solving the problem entails an application of mathematics, but the required mathematics is either directly indicated or obvious.	Solving the problem entails an application of mathematics and requires an interpretation of the context to determine the procedure or concept (may include extraneous information). The mathematics is not immediately obvious. Solving the problem requires students to decide what to do.	In addition to an interpretation of the context, solving the problem requires recognizing important features, and formulating, computing, and interpreting results as part of a modeling process.

Achieve framework cognitive complexity

Solving the problem entails an application of mathematics and requires an interpretation of the context to determine the procedure or concept (may include extraneous information). The mathematics is not immediately obvious. Solving the problem requires students to decide what to do.	In addition to an interpretation of the context, solving the problem requires recognizing important features, and formulating, computing, and interpreting results as part of a modeling process.

Two cautions (Hugh Burkhardt)

- The difference between learned models and autonomous active modeling.
- 'The "few year gap" between the math content that students can do in short exercises and that they can deploy autonomously in modelling new situations.'

What does it mean to take modeling seriously in the high school classroom?

IM high school curriculum

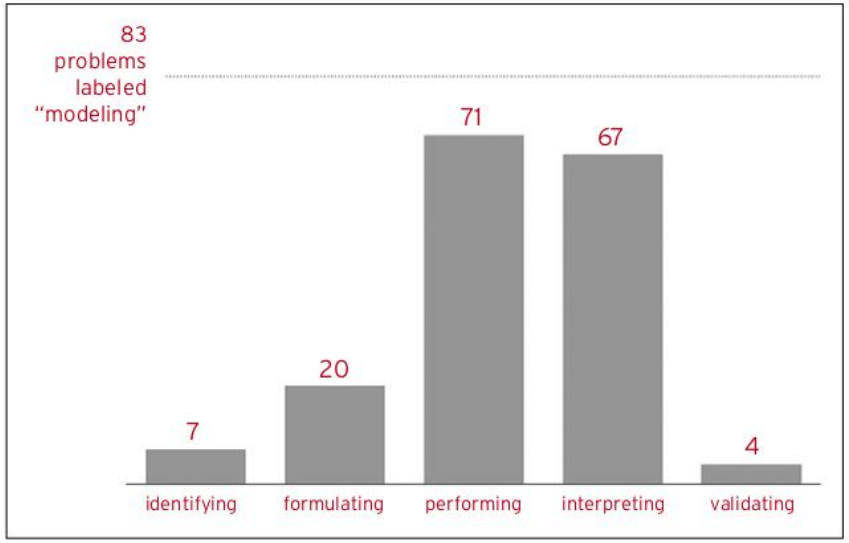
- High school: Algebra 1, Geometry, Algebra 2 and associated teacher professional learning
- Extra Support Materials for Algebra
- Addressing the non-(+) Common Core State Standards
- Available July 1029

Modeling in the IM curriculum

- Many activities address some aspects of the modeling cycle, indicated by the "aspects of mathematical modeling" tag in the materials, including the commonly neglected aspects.
- Modeling prompts provide opportunities outside the lesson plans for students to engage in the complete modeling cycle.

"Missing the Math" (Dan Meyer)

- Identifying essential variables.
- Formulating models.
- Performing operations.
- Interpreting results.
- Validating conclusions.



Mathematics Teacher, 2015

Which aspects of modeling do these problems address?

Heights and handedness

Is there a connection between a student's dominant hand and their size? Use the table of information to compare the size of students with different dominant hands.

handedness	height (cm)	foot length (cm)	arm span (cm)
Left-Handed	173	25	170
Left-Handed	134	65	136
Left-Handed	165	21	168
Left-Handed	180	27	181
Left-Handed	156	23.5	158
Left-Handed	179	25	179
Left-Handed	175	25	170
Left-Handed	189	27	192
Left-Handed	165	21.2	176.8

Pizza party

Imagine your class is having a pizza party.
Work with your group to plan what to order and to estimate what the party would cost.

1. Record your group's plan and cost estimate. What would it take to convince the class to go with your group's plan? Be prepared to explain your reasoning.
2. Write down one or more expressions that show how your group's cost estimate was calculated.
3. a. In your expression(s), are there quantities that might change on the day of the party? Which ones?

   b. Rewrite your expression(s), replacing the quantities that might change with letters. Be sure to specify what the letters represent.

Parade balloon

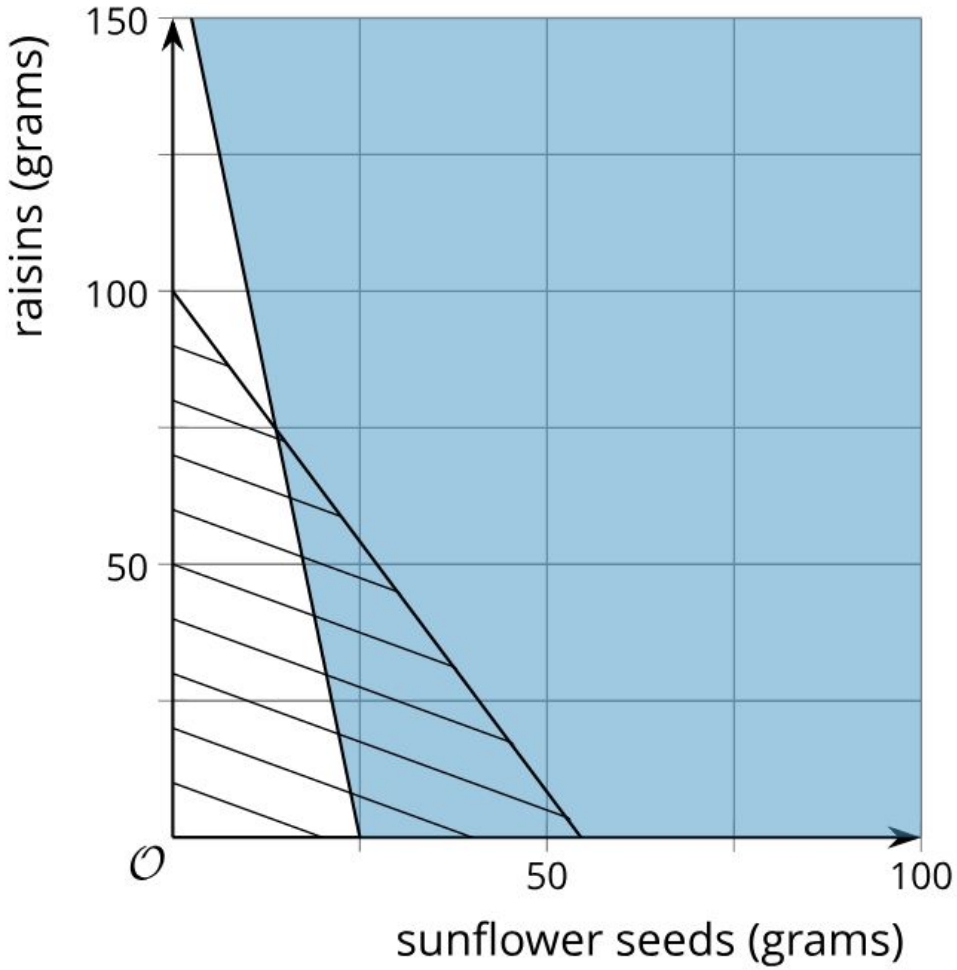
A company makes giant balloons for parades. They're designing a balloon that will be a dilated version of a drum similar to the one in the image. The real-life drum's diameter is 36 inches and it's 1 foot wide.

1. What's the approximate volume of the drum in cubic feet? Round to the nearest hundredth.
2. Suppose the drum is dilated by scale factor k. Write an equation that gives the volume, V, of the dilated drum.
3. What are some reasons the actual drum volume might be different from what you calculated?

Make your own trail mix

	Calories per gram (kcal)	Protein per gram (g)	Sugar per gram (g)	Fat per gram (g)	Fiber per gram (g)
peanuts	5.36	0.21	0.04	0.46	0.07
almonds	5.71	0.18	0.21	0.46	0.07
raisins	3.00	0.03	0.60	0.00	0.05
chocolate pieces	4.76	0.05	0.67	0.19	0.02
shredded coconut	6.67	0.07	0.07	0.67	0.13
sunflower seeds	5.50	0.20	0.03	0.47	0.10
dried cherries	3.25	0.03	0.68	0.00	0.03
walnuts	6.43	0.14	0.04	0.61	0.07

Make your own trail mix

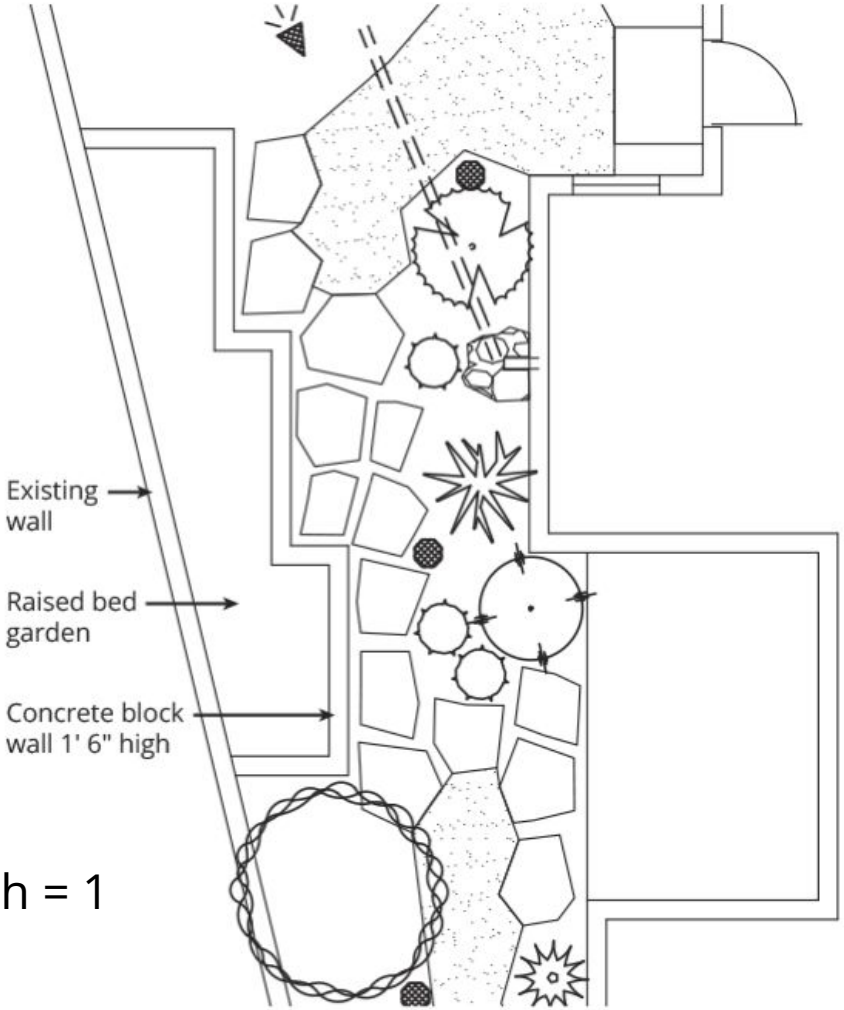


Modeling prompts

- Include multiple versions of a task which require more or fewer aspects of modeling
- Sample solutions, instructions for launching the prompt and supporting students
- Analysis of how heavy the lift is for each version
- Can take one or more days of instruction
- Can be assigned as a project spanning several days or weeks

The garden wall, version 1

A homeowner is going to build the raised bed garden enclosed by a low concrete block wall shown in the plan, up against an existing garden wall and fill the space with soil to grow plants. The homeowner will do all the work and will buy the materials. How much will it cost to make the garden?



Scale: ⅛ inch = 1 foot

The garden wall, version 2

A homeowner is going to build the raised bed garden shown in the plan, against an existing garden wall. The homeowner will build the new wall by stacking concrete blocks without mortar, and then fill in the walled space with garden soil. The concrete blocks have dimensions 8 inches x 8 inches x 16 inches. The wall will be capped by capstones that have dimensions 8 inches x 2 inches x 16 inches. The homeowner will do all the work and will buy materials. How much will they cost?

The garden wall, version 3

A homeowner is going to build the raised bed garden shown in the plan, against an existing garden wall. The homeowner will build the new wall by stacking concrete blocks without mortar, and then fill in the walled space with garden soil. The concrete blocks have dimensions 8 inches x 8 inches x 16 inches. The wall will be capped by capstones that have dimensions 8 inches x 2 inches x 16 inches. Concrete blocks cost $1.50 each, capstones cost $1.00 each, and garden soil costs $36 per cubic yard. The homeowner will do all the work and will buy materials. How much will they cost?

The actual garden wall

Attributes of modeling prompts

- Defining the Question (DQ)
- Quantities of Interest (QI)
- Source of Data (SD)
- Amount of Data Given (AD)
- The Model (M)

attribute	DQ	QI	SD	AD	M	mean
lift	0	1	0	0	2	0.6

Attributes of modeling prompts

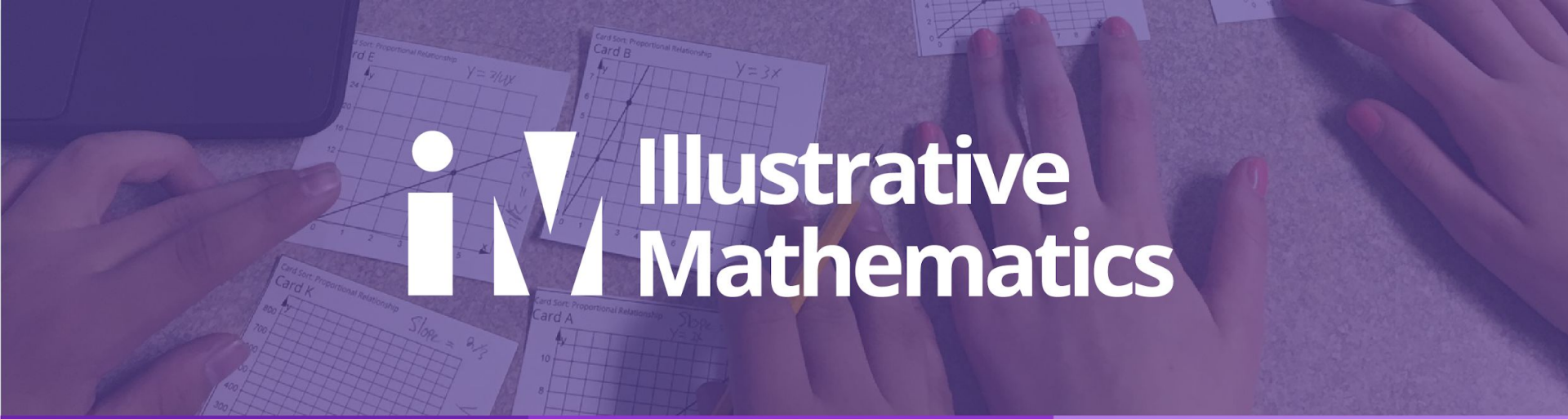
index	attribute	light lift (0)	medium lift (1)	heavy lift (2)
DQ	Defining the Question	well-posed question	elements of ambiguity; prompt might suggest ways assumptions could be made	freedom to specify and simplify the prompt; modeler must state assumptions
QI	Quantities of Interest	key variables are declared	key variables are suggested	key variables are not evident
. . .				

Swing time

1. What are some variables that might affect the period of a pendulum?
2. Collect some data. Which variable appears to have the biggest effect on the period of the pendulum? Justify your response.
3. Create a mathematical model relating the variable you identified to the period of a pendulum.
4. Think carefully about how you decided how many digits to record in your measurements. Explain these decisions.
5. Use your model to determine the characteristics of a pendulum that would have a period of 2 seconds.
6. Would it be possible to create a pendulum with a period of 1 minute? 1 hour? If so, what would you need to create these pendulums? If not, why not?
7. Think carefully about how you decided how many digits to include in the lengths of the 2-second, 1-minute, and 1-hour pendulums. Explain these decisions.

Guidance on modeling

- Each course has a sample prompt with sample student work for students to study.
- There is a handout for students giving advice on modeling.
- There is a rubric for evaluating student work.
- The teacher guide has guidance on introducing modeling to students.



Algebra 1, Geometry, Algebra 2	IM Certified™ 6-8 Math	K–5 Math
Available for fall implementation	Available Version 3.1415 available for fall implementation including Spanish translations and EL supports for teachers	Coming July 2021

Illustrative Mathematics CERTIFIED™

Perfect Score	*Perfect Score*	*Top Rated*
Focus & Coherence	**Rigor and Mathematical Practices**	**Usability**
14/14 **6th, 7th & 8th**	18/18 **6th, 7th & 8th**	37/38 **6th, 7th & 8th**

Scores for the LearnZillion instance of the IM 6-8 curriculum

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